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Slow dynamics due to entropic barriers in the one-dimensional 'descent model'

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Abstract. We propose a novel one-dimensional simple model without disorder exhibiting slow dynamics and aging at the zero temperature limit. This slow dynamics is due to entropic barriers. We exactly solve the statics of the model. We derive an evolution equation for the slow modes of the dynamics which are responsible for the aging. This equation is equivalent to a random walker on the energetic landscape. This latter elementary model can be solved analytically up to some basic approximations and is shown to present aging by itself, as well as a slow logarithmic relaxation of the energy: $\langle e \rangle(t) \sim 1/\ln(t)$ at large t.

PACS. 02.50.Ey Stochastic processes – 05.70.Ln Nonequilibrium thermodynamics, irreversible processes

1 Introduction

Below their glassy transition temperature, glassy systems relax very slowly and remain out of equilibrium on experimental time scales. Despite intensive theoretical studies, the exact nature of this transition is not clearly understood yet, even if it is widely believed that it is of dynamical nature [1]. Glassy systems seem to be trapped in some metastable states whose numbers and life times increase dramatically with decreasing energy. Therefore there is an increasing demand for paradigmatic toy-models containing the elementary physical mechanisms responsible for glassiness and its experimental manifestations, such as the aging effect [2]. Among several other mechanisms for slow relaxation and aging, the focus has recently been brought into the existence of entropic barriers [3]. This term designates boundaries between regions of the phase space where the system has very rare possibility to find a path to go from one region to the other. The system can be trapped in such a region without the necessity of energetic barriers. By analogy with Arrhenius law, the height ΔS of an entropic barrier is defined by $\tau = \exp(\Delta S)$, where τ is the time the system needs to go through the entropic barrier. Simple models [4] – such as the *backgammon* model [5] or *urn* models [6] – have been developed which exhibit such entropic barriers. These models are of mean-field type such as most of models for glassy dynamics. As compared to these simple models for glassiness and entropic barriers, the interaction in the present toy-model is one-dimensional and therefore it can be legitimately considered as more physical.

On the other hand, our model is based on the well-characterized permutation group Σ_n, and many calculations can be carried out either exactly or after some basic assumptions.

2 The model

We consider a one-dimensional system of size n where the configurations are the $n!$ permutations on $\{1, \ldots, n\}$ of the symmetric group Σ_n. We represent a configuration $\sigma \in \Sigma_n$ by a *word* $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ with $\sigma(i) \in \{1, \ldots, n\}$. The energy E of a configuration is defined by its number of descents. One says there is a descent between i and $i+1$ if $\sigma(i) > \sigma(i+1)$. For example, let $n = 6$, the permutation

$$\sigma_{ex} = 15 \downarrow 236 \downarrow 4 \tag{1}$$

has 2 descents symbolized by an arrow $\downarrow$ and its energy is therefore $E(\sigma_{ex}) = 2$.

We denote by D_n^k the degeneracy of the energy level $E = k$. The D_n^k are known as Euler numbers [7]. The identity is the unique ground state with zero energy so that $D_n^0 = 1$. Furthermore we can define a symmetric permutation σ^c for all $\sigma \in \Sigma_n$, by

$$\sigma^c = n + 1 - \sigma(i), \tag{2}$$

for $1 \leq i \leq n$, which has the energy $E(\sigma^c) = n - 1 - E(\sigma)$. The distribution of configurations with energy k is symmetric with respect to $(n-1)/2$, and $D_n^k = D_n^{n-1-k}$. At high temperatures all configurations have the same probability, the mean energy per particle in this limit is

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$\langle e\rangle = (n-1)/2n$, according to the above symmetry and $\langle e\rangle = 1/2$ at the large size limit.

The system evolves *via* a Metropolis Monte-Carlo algorithm [8] and we study two different dynamics: in the non-local one, at each time step, any two sites $i \neq j$ are chosen at random and we try to transpose the elements $\sigma(i), \sigma(j)$ with a certain probability P_{ji}, which depends on the temperature T and on the energy variation ΔE if the transposition were executed: $P_{ji} = \min(1, \exp(-\Delta E/T))$. In the local dynamics we choose only one site i at random, and we try to transpose $\sigma(i)$ and $\sigma(i+1)$, with the same transition probability as above. Sometimes in the following we will call *particles* the $\sigma(i)$. Indeed, this model can be seen as a system of n distinguishable labelled particles on a one-dimensional lattice which tend to sort themselves.

We anticipate on the following to emphasize the origin of the entropic barriers in the non-local dynamics. If *quenched* from infinite to vanishing temperature the system evacuates its energy and relaxes through decreasing energy levels towards its equilibrium state. We shall demonstrate in the following that it encounters entropic barriers between any two successive energy levels whereas it relaxes rapidly inside each energy level. For example let us explain why there is an entropic barrier between the first excited states and the unique fundamental one. We need to know more precisely the values of D_n^k. Starting from the expression [7]:

$$(k+1)^n = \sum_{j=0}^{k} D_n^k \binom{n+k-j}{n}, \tag{3}$$

we derive the value of $D_n^1 = 2^n - (n+1) \simeq 2^n$ for $n \gg 1$. We get in the same way:

$$D_n^k \simeq (k+1)^n, \text{ for } k/n \ll 1. \tag{4}$$

We can easily recover the result for D_n^1 by constructing the permutations with one descent as follows: we place the elements of the permutation, beginning by 1 and continuing with 2 etc., randomly in two different sets, like if we were distributing distinguishable particles into two boxes. We can build 2^n permutations that have a descent at the boundary of the two sets, except if we have built the identity, which is obtained in $n+1$ ways, moving this boundary between any two successive elements of the identity. Subsequently $D_n^1 = 2^n - (n+1)$. This point of view can be generalized to any energy k. The system is seen as $(k+1)$ ordered subsets separated by descents. In this point of view, the descents can be considered as domain walls between ordered subsets.

This point of view also highlights the similitude between our model and the *backgammon* [5,9] one. It consists in N distinguishable particles placed in N boxes, where the energy is equal to minus the number of empty boxes. Our subsets play the role of the boxes in the backgammon model. The difference with our model is that the particle interaction in the backgammon model is not localized: it is intrinsically mean-field. Note however that there exists a one-dimensional generalization of the backgammon model [10] where particles are allowed to jump to nearest neighbors only. However, this generalization complicates the analysis of the model [10].

We have seen above that there are about 2^n first excited states and only one fundamental. In order to find this state, the system with energy equal to one, will wander in the phase space until it finds a state connected to the fundamental one. If the wandering is uniform in the first level, the probability that the system finds such a state, will be $(n(n-1)/2)/2^n$, where $n(n-1)/2$ is the number of path starting from the fundamental state in the non-local dynamics. Being in one of these states, it will have a probability $2/n(n-1)$ to choose the direction towards the minimal energy. So, there will be about 2^n steps of the dynamics before the system finds the identity state, which corresponds to an entropic barrier of height of order n. Note that the previous argument also holds in the local dynamics: one obtains the same entropic barrier. Nevertheless, the type of the dynamics will be of importance, since one can suppose an uniform wandering only for the non-local one for which we can prove the connectivity of the energy levels even at $T = 0$. Which means that there is always a path between any two configurations of one level that is inside it. The system searching the paths to decrease its energy will not have to pass through any energy barrier. It is not the case for the local dynamics. But the entropic barriers, which depend mainly on the ratio of the level sizes, will be also present in the local dynamics, in conjunction with energetic barriers. In terms of permutation, passing through an entropic barrier corresponds to a complex rearrangement of particle. The order in which pairs of particles must be swapped to optimize their position without increasing the energy is quite constrained.

3 Statics

One of the advantages of the model is that its partition functions, canonical or grand canonical, have been extensively studied in the mathematical literature. Indeed, we can write the canonical partition function as follows:

$$Z = \sum_{k=0}^{n-1} D_n^k \exp\left(-\frac{k}{T}\right) = \sum_{k=0}^{n-1} D_n^k t^k = \frac{1}{t}(1-t)^{n+1} \sum_{k\geq 0} k^n t^k, \tag{5}$$

with $t = \exp(-1/T)$ and where the last equality comes from combinatorial analysis [11]. More precisely, it is obtained inductively using the following relation on the D_n^k:

$$D_{n+1}^k = (n+1-k) D_n^{k-1} + (k+1) D_n^k. \tag{6}$$

This former equation is obtained by building configurations with $n+1$ elements by adding the $(n+1)$th particle in all the possible positions on a configuration with n elements. One can further remark that equation (4) comes from the second term of the right hand-side of equation (6). To obtain the mean values of the thermodynamical observables we compute:

$$\frac{\ln Z}{n} = \frac{n+1}{n} \ln(1-t) + \frac{1}{n} \ln \sum_{k\geq 0} k^n t^{k-1}. \tag{7}$$

The sum in the right hand side of the former equation reads

$$\sum_{k\geq 0} k^n t^{k-1} = \sum_{k\geq 0} \exp -n(\ln(k+1) - \frac{k}{nT}) \qquad (8)$$
$$= \sum_{k\geq 0} \exp(-nf(k,T))$$
$$= (2\pi n T^2)^{1/2} \exp(1/T - n),$$

by taking the continuous limit, and evaluating the so-obtained integral by a saddle-point argument for $n \gg 1$. This continuous limit is only valid if the width of the Gaussian approximate of $\exp(-nf(k,T))$, is large as compared to the spacing between two consecutive energy levels, $\Delta k = 1$, that is to say if $T \gg 1/2\sqrt{n}$. In the thermodynamic limit it will be always true for any finite temperature. Using equation (7) and thermostatics identities the thermodynamical observables can be exactly computed. The free energy per site obeys

$$f = -T(\ln(1-t) + \ln(nT)) \qquad (9)$$

and the mean energy per particle is given by

$$\langle e \rangle = T - t/(1-t). \qquad (10)$$

It follows that the specific heat is

$$C_V = 1 + t/(T(1-t))^2, \qquad (11)$$

and the entropy per particle is given by

$$s = (\langle e \rangle - f)/T = \ln(nT) + \ln(1-t) - t/T(1-t) + 1. \qquad (12)$$

Note that this entropy is not extensive. Let us remark the existence of an interesting low temperature domain defined typically by $T < 0.1$, where the first term of equation (7) is negligible so that $\langle e \rangle \simeq T$. It delimits a low energy domain, where $D_n^k \simeq (k+1)^n$, which results in entropic barriers. We will see in the following part that the slow dynamics takes place in this region of the configuration space. These results show that there is no thermodynamical transition at any finite temperature.

4 Dynamics

We have defined two types of dynamics above, for the sake of simplicity, we focus on the non-local one, and we shall discuss the local one in the conclusion. As we have seen above, we study the dynamics of the system after a *quench* from high temperature, during which the system tries to decrease its energy to reach a low temperature equilibrium state. In the following, we will focus only on low temperatures such that the system has to encounter entropic barriers during its relaxation process.

In order to analyse the dynamics, we first map our model on a random walker on the energy levels. It is on this model, which is much simpler, that we get analytical results. In a second part, we check numerically by Monte-Carlo simulations that the dynamics of the *descent* model is quite well described by this random walker. In order to justify this mapping let us make some hypotheses. We suppose that the equilibration time in any energy level is short as compared to the time spent to go through an entropic barrier between two energy levels. That is to say we suppose that there is no entropic barrier inside an energy level. This hypothesis is reasonable since the energy levels are connected, and will be corroborated in the following by numerical results. We also suppose that the paths between two energy levels are uniformly distributed inside these levels.

Following these hypotheses we compute the probability that the system goes from one level to another using the mean features of the levels. We take the slow dynamics of the model as equivalent to a dynamics between the energy levels. In other words, we map the dynamics of our model on a Markovian random walker on the energy levels, whose features are precised in the following. Since we know the D_n^k at low energy, and the number of paths starting from any configuration, we know the total number of paths starting from one level, $D_n^k(n-1)n/2$. We are searching the number of paths allowing the system to decrease its energy from one level k. It is also the number of paths starting from one level $l < k$ and arriving in k. Since the energy variation cannot exceed 2, $l \in \{k-2, k-1\}$. Let us introduce a parameter $p(k)$ which represents the proportion of paths starting from the level $k-1$ and allowing the system to *increase* its energy by one unit. It certainly depends on the energy level. We investigated numerically the configuration space of the model, and we concluded that for $\frac{k}{n} = e < 0.1$, $p(k) \simeq Ke$, with K a constant that seems to depend very weakly on n. For $n = 10000$, we have $K \simeq 1.99$ and $K = 1.97$ for $n = 12$. For the links between the fundamental state and the first excited ones ($k = 1$), one can trivially show that K exactly equals 2. In the following we will consider $K = 2$.

So, $pD_n^{k-1}(n-1)n/2$ is the number of paths starting from one level $k-1$ and *increasing* the energy by one unit, and pD_n^{k-1}/D_n^k the proportion of paths starting from one level k allowing the system to *decrease* its energy by one unit. Being equilibrated in the level k, the system has to find one of these paths to decrease its energy. These events are very rare because of the great ratio D_n^k/D_n^{k-1} at low energy. Indeed, the typical time $\tau_{k,1}$, in Monte-Carlo steps, to go through the entropic barrier (of height ΔS) between the energy levels k and $k-1$, satisfies:

$$\tau_{k,1} = \exp(\Delta S) = \frac{D_n^k}{pD_n^{k-1}} \simeq \frac{1}{p}\left(\frac{k+1}{k}\right)^n \overset{n\gg 1}{=} \frac{1}{p}\exp(n/k). \qquad (13)$$

This time gives us the probability rate by unit step that the random walker goes from k to $k-1$, $\omega_{k\to k-1} = 1/\tau_{k,1}$. In the same way we can obtain the typical time $\tau_{k,2} \simeq \exp(2/(k/n - 1/n))/q$ that the system goes from a level k to $k-2$, q being the proportion of paths starting from k and arriving in $k+2$. Simulations show that q is of order 1. We obtain $\omega_{k\to k-2} = 1/\tau_{k,2}$, but in the following we will always neglect it since $\omega_{k\to k-2}/\omega_{k\to k-1} \simeq \omega_{k\to k-1} \ll 1$.

The master equation for the random walker at $T = 0$ is given by:

$$P(k, t + 1/n) = P(k, t) + p(k + 1) \exp(-n/(k + 1))P(k + 1, t) - p(k) \exp(-n/k)P(k, t), \quad (14)$$

where t is in Monte-Carlo unit (one unit is equal to n Monte-Carlo steps). By using the energy per particle e and the continuous limit in time and energy, this equation reads:

$$\frac{\partial P(e,t)}{\partial t} = \frac{\partial}{\partial e}(p(e)P(e,t) \exp(-1/e)). \quad (15)$$

Thus one obtains the evolution of the mean energy per particle with time $\langle e\rangle(t)$, using $p(e) = 2e$:

$$\frac{d\langle e\rangle}{dt} = \int_0^{(n-1)/n} \frac{\partial P(e,t)}{\partial t} e \, de = -2 \int_0^{(n-1)/n} eP(e,t) \exp(-1/e) de. \quad (16)$$

We calculate this integral by developing $eP(e, t) \exp(-1/e)$ around $\langle e\rangle$:

$$\frac{d\langle e\rangle}{dt} = -2\langle e\rangle \exp(-1/\langle e\rangle) + \mathcal{O}(\Delta e^2 \exp(-1/\langle e\rangle)/\langle e\rangle^3), \quad (17)$$

where one keeps only the zeroth order in $e - \langle e\rangle$, since we shall demonstrate that $\Delta e^2 = \langle (e - \langle e\rangle)^2\rangle \simeq \langle e\rangle^2/2n$. Indeed we can derive time evolution of Δe^2 from equation (14) paying attention to keep terms of order $1/n$ since they cannot be neglected in this case. As above, discrete sums are evaluated using an integral and developing $eP(e, t) \exp(-1/e)$ around $\langle e\rangle$:

$$\frac{d\Delta e^2}{dt} = -\frac{2}{\langle e\rangle} e^{-1/\langle e\rangle} \Delta e^2 + \frac{\langle e\rangle}{n} e^{-1/\langle e\rangle} - \frac{1}{n} e^{-2/\langle e\rangle}. \quad (18)$$

The first term describes a rapid decay of Δe^2 towards its quasi-equilibrium value $\langle e\rangle^2/2n$. Hence Δe^2 will be stabilized around $\langle e\rangle^2/2n$, which is checked numerically for the *descent* model and the random walker. By integrating equation (17) we obtain the time the walker needs to go from high energy to the level $\epsilon = \langle e\rangle$, $t = \epsilon \exp(1/\epsilon)/2 + \mathcal{O}(\epsilon^2 \exp(1/\epsilon))$ and:

$$\langle e\rangle(t) \simeq \frac{1}{\ln(2t) + \ln(\ln(2t))} \overset{t\to\infty}{\sim} \frac{1}{\ln(t)}. \quad (19)$$

This dynamics can also be found in a variety of theoretical models [4], such as the *backgammon* model and the *oscillator* model [12], as well as in compaction of granular media (models and experiments [13–15]; see conclusion). This result means that the system will never reach the fundamental state at $T = 0$, and will always stay out of equilibrium despite the absence of energy barriers. Its relaxation time in the thermodynamical limit is diverging. In the long time regime the dynamics will be slower and slower. To check equation (19), we made some numerical simulations on the random walker and on the original model. The interest of the random walker is that its simplicity allows us to compute numerically its *exact* dynamics for large n and t (up to $t = 10^{12}$) using its transition matrix. We find very good agreement between analytical and numerical calculations. Monte-Carlo simulations on the descent model are also in very good agreement with analytical calculations at large t, as we can see in Figure 2. These results corroborate *a posteriori* the hypothesis that there are no entropic barriers inside the energy levels, since otherwise the random walker would be faster than the *descent* model.

Let us remark that the only *analytically unknown parameter* is $p(e)$. In fact, this parameter is irrelevant for the asymptotic dynamics while p is of polynomial type: $p = Ke^\alpha$. Indeed, using this equation for p, equation (17) reads:

$$\frac{d\langle e\rangle}{dt} = -K\langle e\rangle^\alpha \exp(-1/\langle e\rangle) + \mathcal{O}(\Delta e^2 \exp(-1/\langle e\rangle)\langle e\rangle^{\alpha-4}), \quad (20)$$

and hence the mean energy per particle becomes:

$$\langle e\rangle(t) \simeq \frac{1}{\ln(Kt) + (2 - \alpha)\ln(\ln(Kt))} \overset{t\to\infty}{\sim} \frac{1}{\ln(t)}. \quad (21)$$

For $\alpha = 2$, the evolution in $1/\ln(t)$ is even exact at all times in the thermodynamical limit, not only asymptotically, which is well verified in numerical simulations.

The variance Δe^2 does not depend on α, since equation (18) becomes:

$$\frac{d\Delta e^2}{dt} = -2\langle e\rangle^{\alpha-2} e^{-1/\langle e\rangle} \Delta e^2 + \frac{\langle e\rangle^\alpha}{n} e^{-1/\langle e\rangle}. \quad (22)$$

We still have the quasi-equilibrium value $\Delta e^2 = \langle e\rangle^2/2n$. We will see below that the shape of p does not change the aging in the energy-energy correlation function either. It is not surprising: what dominates the dynamics is the exponential decay of the random walker transition rate with the inverse energy.

In the following we will consider again $p = 2e$, since it mimics the best the *descent* model.

To study the dynamics at $T > 0$, we have to consider the possibility that the walker increases its energy. From the definitions of the dynamics and of p and q, it follows that $\omega_{k\to k+1} = p \exp(-\beta)$ and $\omega_{k\to k+2} = q \exp(-2\beta)$, where β is the inverse temperature. At low temperature, $\omega_{k\to k+1}/\omega_{k\to k+2} \ll 1$, so we need not consider the possibility that the system jumps two levels in one step. In the same way as at $T = 0$, the evolution of the random walker position follows a master equation, which in the continuous limit, leads to:

$$\frac{d\langle e\rangle}{dt} = -2\langle e\rangle \exp(-\frac{1}{\langle e\rangle}) + 2\langle e\rangle \exp(-\beta) + \mathcal{O}(\Delta e^2 \exp(-1/\langle e\rangle)/\langle e\rangle^4). \quad (23)$$

We remark that $d\langle e\rangle/dt = 0$ for $\langle e\rangle = 1/\beta = T$, which is the equilibrium energy for the *descent* model for

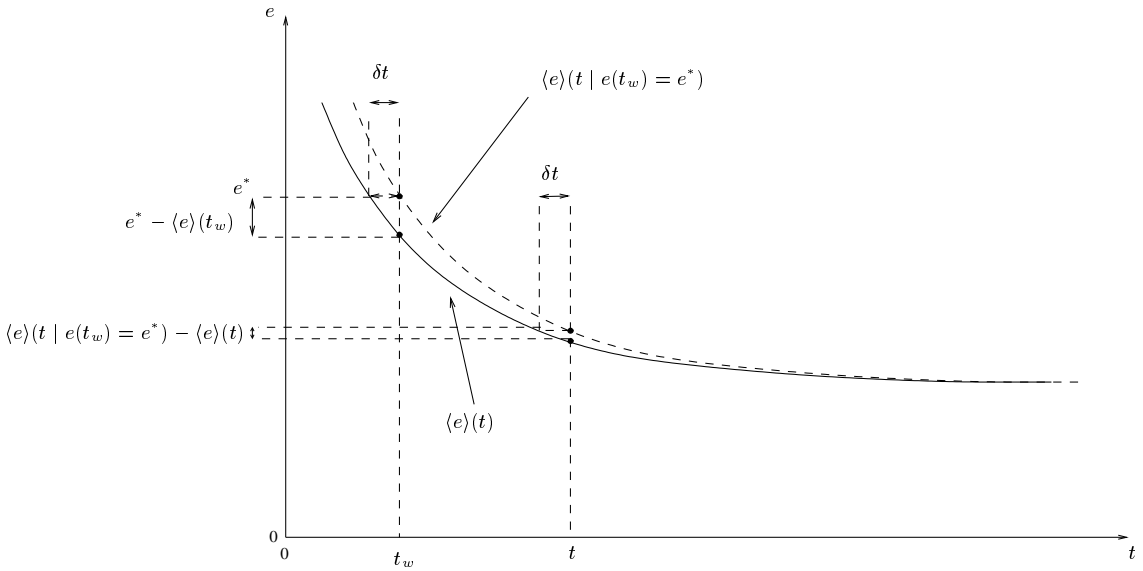

Fig. 1. Schematic picture of the aging analysis. The full line represents the mean energy per particle whereas the dashed line represents the mean energy per particle when the system is at energy e^* at time t_w. δt is the constant time shift between these two curves.

$T \ll 1$. At high T the random walker does not represent the *descent* model anymore since it evolves in a region where $D_n^k \neq (k+1)^n$. Thus we focus on the low T region. One can see from equation (23), that while the system is far from its equilibrium energy the possibility that it jumps to an higher level by thermal activation will be negligible in front of its possibility to decrease its energy: for $\langle e\rangle \gg 1/\beta$ we have $\exp(-\frac{1}{\langle e\rangle}) \gg \exp(-\beta)$. Far from its equilibrium state, the system is not influenced by the temperature and its dynamics is the same as at $T = 0$. This defines a typical time $\tau_\beta = \exp(\beta)/(2\beta)$ needed by the system to reach $\langle e\rangle = 1/\beta$ with the dynamics at $T = 0$. Near equilibrium, temperature effects appear and we obtain the dynamics by developing $\exp(-1/\langle e\rangle)$ around $1/\beta$ in equation (23). We find an exponential relaxation of energy with the same characteristic time τ_β (see Fig. 2, inset).

We define two types of correlation functions in order to characterize the dynamics. One is based on the matching of two permutations, in others words on the proportion of sites at time t which bear the same particles as at a time t_w: $C_\sigma(t, t_w) = \sum_1^n \delta_{\sigma_i(t+t_w),\sigma_i(t_w)}/n$, where t_w is the waiting time after the quench. At high temperature, it can be proven that $C_\sigma(t \gg t_w, t_w) = 1/n$, at lower temperature we numerically observe that it is also the case. This correlation function shows rapid decay for any t_w, with no aging effects. The second correlation function we consider is the energy-energy correlation:

$$C_e(t, t_w) = \frac{\langle e(t+t_w)e(t_w)\rangle - \langle e(t+t_w)\rangle\langle e(t_w)\rangle}{\Delta e^2(t_w)}. \quad (24)$$

We studied it numerically for both models at different t_w. For the random walker, at $T = 0$ one can see in Figure 3 that the system ages. The correlation function seems to tend towards the scaling law: $C_e(t, t_w) \simeq t_w/(t + t_w)$ at very large t_w.

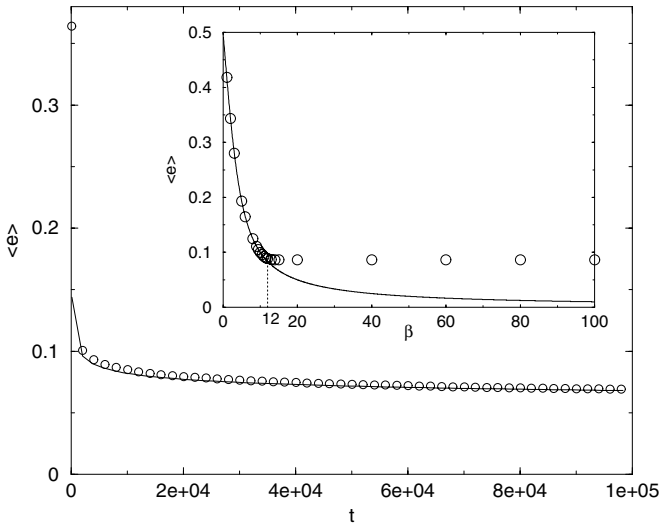

Fig. 2. $\langle e\rangle(t)$ for $n = 500$ and $T = 0$. Circles represent Monte-Carlo simulations of the *descent* model compared with the evolution given by $1/(\ln(Kt) + \ln(\ln(Kt)))$ (full line) which represents the energy of the random walker with $p = Ke$, $K = 2$. Circles in the inset show $\langle e\rangle$ for the *descent* model at a fixed time $t = 10^4 = \tau_{(\beta=12)}$ for various temperatures and the full line the exact static energy. We check that for β such that $\tau_\beta < 10^4$ the system has reached its equilibrium energy, and that for all β such that $\tau_\beta > 10^4$ the system is out-of-equilibrium and its energy is independent of β.

At $T > 0$ we expect the dynamics to be the same as at $T = 0$ while the system is out-of-equilibrium. As long

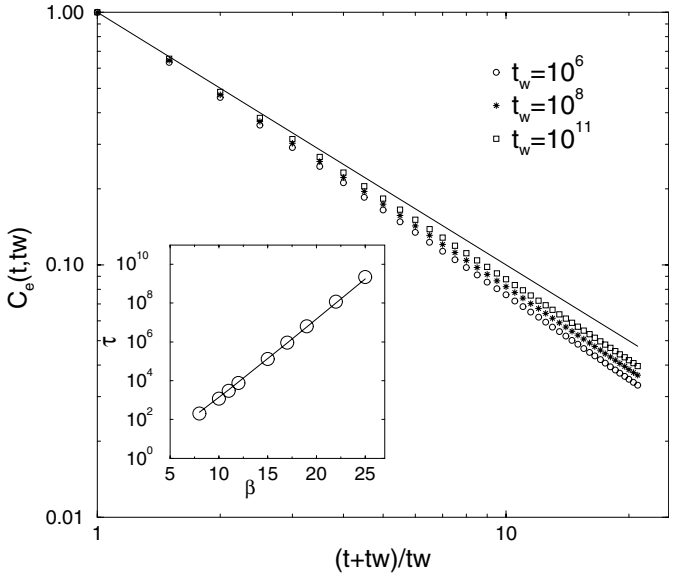

Fig. 3. Energy-energy correlation of the random walker at $T = 0$, for $n = 2000$, as a function of $(t+t_w)/t_w$, compared with $t_w/(t + t_w)$ (full line). The inset shows the relaxation time as a function of β, of the random walker, obtained from $C_e(t, t_w)$ at equilibrium, and the full line a fit by $A\exp(\beta)/2\beta = A\tau_\beta$, with $A = 1.25$, corroborating analytical calculations.

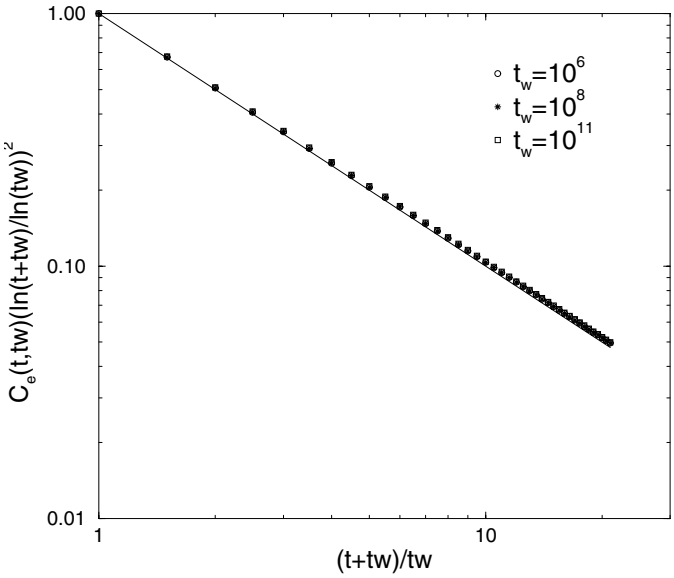

Fig. 4. Energy-energy correlation corrected by $\ln^2(t + t_w)/\ln^2(t_w)$ of the random walker at $T = 0$, for $n = 2000$, as a function of $(t + t_w)/t_w$, compared with $t_w/(t + t_w)$ (full line). The logarithmic corrections ensure the collapse of the data for different values of t_w.

as $t < \tau_\beta$ we find numerically the same plots as at $T = 0$ whereas we find time translational invariance for $t + t_w > \tau_\beta$. The same results are found for the *descent* model but with statistical noise since we cannot use the transition matrix. These results show the strong importance of the observable we focus on to observe aging [17].

Now we propose a simple argument leading to the following law at large t and t_w :

$$C_e(t, t_w) \simeq \frac{t_w}{t + t_w} \frac{\ln^2 t_w}{\ln^2(t + t_w)}. \tag{25}$$

As illustrated in Figure 4, this law accounts quite well for our numerical observations as far as the random walker is concerned. In particular, the logarithmic corrections ensure the collapse of the curves for different values of t_w on a straight line of slope very close to -1. For the *descent* model, the statistical noise does not allow to distinguish if the logarithmic corrections improve the agreement between theory and numerical experiments.

Our argument is as follows: since the dynamics on energy levels is Markovian, the future of the random walker beyond t_w (*i.e.* $t > 0$) only depends on the energy distribution at time t_w. For the *descent* model it will be only true at large times, when the entropic barriers make the dynamics on the energy levels Markovian.

Now we make the following hypothesis: the evolution of the mean energy $\langle e\rangle(t+t_w)$ only depends on the mean energy $\langle e\rangle(t_w)$ and not on the precise shape and width of the energy distribution at t_w. This hypothesis is supported by numerical simulations with different initial energy distributions and it is corroborated by the excellent agreement between theoretical and numerical correlation functions. Furthermore, it amounts to neglecting the terms of higher order in equation (17), as we made before, which is exact in the thermodynamics limit.

A consequence of this hypothesis is that any shift δe at t_w of the mean energy has the same effect on $\langle e\rangle(t+t_w)$ as a time delay δt at t_w, since it only consists of a shift of the initial condition. For small δe, δt is such that:

$$\frac{\delta e}{\delta t} \simeq \frac{d\langle e\rangle}{dt}(t_w) = -\frac{1}{t_w \ln^2 t_w}. \tag{26}$$

This point is illustrated in Figure 1. Now we still denote by $\langle e\rangle(t + t_w)$ the mean energy without any shift at t_w, and by $\langle e|\delta e, t_w\rangle(t + t_w)$ the mean energy after an energy shift δe at t_w (conditional mean). Subsequently

$$\begin{aligned}\langle e|\delta e, t_w\rangle(t + t_w) &\simeq \langle e\rangle(t + t_w + \delta t) \\ &\simeq \langle e\rangle(t + t_w) + \delta t \, \frac{d\langle e\rangle}{dt}(t + t_w) \\ &\simeq \langle e\rangle(t + t_w) + \delta e \, \frac{t_w}{t + t_w} \, \frac{\ln^2 t_w}{\ln^2(t + t_w)}.\end{aligned} \tag{27}$$

Now the energy-energy correlation function is

$$\langle e(t+t_w)e(t_w)\rangle = \sum_{e^*} \sum_{e} e \; e^* \; P(e^*, t_w) P(e, t+t_w|e^*, t_w) \tag{28}$$

where $P(e^*, t_w)$ is the probability that the energy per particle is e^* at time t_w and $P(e, t+t_w|e^*, t_w)$ is a conditional probability, and

$$\begin{aligned}
&\langle e(t+t_w)e(t_w)\rangle \qquad (29)\\
&= \sum_{e^*} e^* P(e^*, t_w) \langle e|\delta e = e^* - \langle e\rangle(t_w), t_w\rangle\\
&= \sum_{e^*} e^* P(e^*, t_w)\\
&\quad \times \left[\langle e\rangle(t+t_w) + (e^* - \langle e\rangle(t_w))\ \frac{t_w}{t+t_w}\ \frac{\ln^2 t_w}{\ln^2(t+t_w)}\right]\\
&= \langle e\rangle(t+t_w)\langle e\rangle(t_w)\\
&\quad + \frac{t_w}{t+t_w}\frac{\ln^2 t_w}{\ln^2(t+t_w)} \sum_{e^*} P(e^*, t_w)\left[e^* - \langle e\rangle(t_w)\right]^2 .
\end{aligned}$$

The last sum is the variance Δe^2 of the energy distribution at time t_w. Hence we get equation (25). Note that the same kind of argument can be used to derive the same energy-energy correlation function for the Backgammon model and is also in excellent agreement with numerical simulations that we performed independently. Note that our correlation function is not the same as the one calculated in reference [16], which explains that is does not have the same expression. With $p = Ke^\alpha$, this law is unchanged as we saw before for $\langle e\rangle(t)$.

5 Conclusion

We have focused so far on the non-local dynamics. Let us now discuss briefly the local one. Its interest lies in the fact that it is truly one-dimensional since it respects the one-dimensional character of the model. We shall see that even if its analysis is complicated by the existence of energetic barriers (which freeze the dynamics at $T = 0$), the qualitative conclusions are the same as for the non-local case. Indeed, one shows numerically that below an energy per particle $e_m(n) \approx 0.1$, which depends slowly on n, nearly all states are local potential minima. Therefore at $T = 0$, the system is always stuck in these minima, and at $T > 0$ it has to pass over energy barriers at each energy level in order to lower its energy. However, these energy barriers are always of height 1, and one needs a time of order $\exp(\beta)$ to pass them. Therefore, the times needed to pass over *energetic* barriers and *entropic* barriers have the same order of magnitude and the system is not substantially slowed by energetic barriers, at least at $T > 0$. More precisely, we have measured typical times $\tau'(n, T)$ by evaluating the constant diffusion of particles at equilibrium, at small but finite temperatures and we conclude that $\tau'(n, T)$ is of dominant order $n^2 \exp(\beta)$. As compared to the non-local characteristic times, this dynamics is n^2 times slower than the non-local one. Note that this prefactor n^2 is also present at high temperatures where it can be proven rigorously that $\tau'(n, \infty) \sim n^2$ [18]. At high temperature, the n^2 term is certainly due to the diffusion of particles over the whole system. At smaller temperature, one can think that it results from the diffusion of descents acting as walls between ordered domains, which allows the system to explore energy levels. As far as aging is concerned, our numericals results remain compatible with the law: $C_e(t, t_w) = t_w/(t + t_w)$.

A natural continuation of the present work will be to investigate into deeper detail the relationship between our model and compaction of granular media. Indeed, equation (17) also governs the evolution of the density in simple models of compaction (see [14] for example). On the other hand, in these *mean-field* models based on a free volume argument, as well as in the *one-dimensional* descent model, the slow dynamics is due to the necessity of complex and long rearrangements of particles to optimize the organization of the system.

To finish with, we mention that as soon as a model presents an effective Markovian dynamics between energy levels due to entropic barriers, the present analysis can be applied to this model. For example, the Backgammon model can be tackled in such a way and one obtains the correct laws for the mean energy $\langle e\rangle(t)$ and the correlation function $C_e(t, t_w)$.

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References

1. W. Kob, Lecture notes for *"Slow relaxations and nonequilibrium dynamics in condensed matter"*, Les Houches, Vol. 77 (EDP Sciences, Springer-Verlag, 2003), pp. 199–270
2. J.P. Bouchaud, L.F. Cugliandolo, J. Kurchan, M. Mézard, *Spin-glasses and Random field*, edited by A.P. Young (World Scientific, Singapore, 1997)
3. C. Godrèche, J.P. Bouchaud, M. Mézard, J. Phys. A **28**, L603 (1995)
4. F. Ritort, P. Sollich Adv. Phys. **52**, 219 (2002)
5. F. Ritort, Phys. Rev. Lett. **75**, 1190 (1995)
6. C. Godrèche, J.M. Luck, J. Phys.: Condens. Matter **14**, 1601 (2002)
7. L. Euler, *Institutionnes calculi differentialis* (Saint-Petersbourg, 1755)
8. M.E.J. Newman, G.T. Barkema, *Monte Carlo Methods in Statistical Physics* (Clarendon Press, Oxford, 1999)
9. S. Franz, F. Ritort, J. Phys. A **30**, L359 (1997)
10. A. Crisanti, F. Ritort, A. Rocco, M. Sellitto, J. Chem. Phys. **113**, 10615 (2000)
11. L. Comtet, *Analyse combinatoire*, T. 2 (Presses Universitaires de France, 1970)
12. L.L. Bonilla, F.G. Padilla, F. Ritort, Physica A **250**, 315 (1998)
13. T. Boutreux, P.G. de Gennes, Physica A **244**, 59 (1997)
14. E. Ben-Naim et al., Physica D **123**, 380 (1998)
15. J.B. Knight et al., Phys. Rev. E **51**, 3957 (1995)
16. C. Godrèche, J.M. Luck, J. Phys. A **29**, (1996) 1915.
17. A. Pérez-Madrid, D. Reguera, J.M. Rubí, J. Phys.: Condens. Matter **14**, 1651 (2002)
18. D.B. Wilson, Ann. Appl. Probab. **14**, 274 (2004)